DIFFERENTIAL APPROXIMATION AS A QUALITATIVE TEST FOR DIFFERENCE SCHEMES

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It is usual to pass from the discrete to the continuous limit in constructing models for applications. Such models are often very complicated for direct examination, and numerical methods are often the only ones to use. A numerical algorithm involves passing from the analog representation to a discrete one. The resulting discrete model must be as close as possible to the initial analog model. The finite-difference method is one of the most widely used. The differential equations are replaced by a difference scheme, which may be constructed in various ways. This makes the importance of qualitative examination of such schemes clear.

Differential approximation is widely used in examining difference schemes for hyperbolic equations [1], which enables one to construct new difference schemes with preset properties, while also enabling one to analyze existing or new difference schemes and to classify such schemes on the properties.

It is shown here that differential approximations can be used in asymptotic analysis of a difference scheme, and it is also shown that invariant difference schemes are closer to the initial analog model than are schemes that are not invariant under analog transformations allowed by the initial differential equations.

1. Asymptotic Expansion of the Solution to a Cauchy Difference Problem. Asymptotic analysis is one of the major methods of qualitative examination for difference schemes. Studies have been made [2-5] of the scope for examining asymptotic behavior in difference schemes by differential approximation. It has been shown [2] that the solutions to some schemes in generalized-function space converge to the corresponding differential approximations for $t \to \infty$ and a given time step. A study has been made [3, 4] of the asymptotic behavior of the solutions to a one-parameter family of difference schemes; the solution to the Cauchy problem for the differential approximation gives an asymptotic representation of the solution to the Cauchy difference scheme in a certain topological space (subject to fairly rigid constraints on the schemes and differential approximations). Following [5], it has also been shown that a similar assertion applies in the linear topological space of generalized functions Z' without substantial restrictions on the difference schemes and differential approximations.

1. For simplicity, we consider the case of one independent spatial variable, although all results apply in the multidimensional case.

We recall certain definitions and symbols from the theory of generalized functions required subsequently.

Let S be the space of rapidly decreasing functions of real argument, while D(R) is the space of finite functions whose carrier is contained on the real axis R, while Z is the space of integer analytic functions of complex argument that satisfy the following condition: for any function $\varphi(z) \in Z$ and for any $k$ ($k \geqslant 0$) there exist constants $a$ and $C_k$ such that

$$|z^k\varphi(z)| \leqslant C_k \exp\{a|\operatorname{Im} z|\}.$$

Generalized-function spaces (spaces of linear continuous functionals) upon the spaces S, D(R), and Z are denoted, respectively, by S', D'(R), Z'.

If f is the generalized function, then the value on a basic function is denoted as (f, φ); if f(x) is a summable function, then $(f, \varphi) = \int\limits_{-\infty}^{+\infty} f(x)\varphi(x)\,dx$.

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The following operators are defined on the above basic-function spaces:

$$F[\varphi](\xi) = \int_{-\infty}^{+\infty} e^{ix\xi}\varphi(x)\,dx,$$

$$F^{-1}[\varphi](\xi) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-ix\xi}\varphi(x)\,dx, \quad D^m\varphi(x) = \frac{\partial^m\varphi}{\partial x^m}, \quad T_y\varphi(x) = \varphi(x+y), \tag{1.1}$$

$$(\varphi * \psi)(x) = \int_{-\infty}^{+\infty} \varphi(y)\psi(x-y)\,dy, \quad \check{\varphi}(x) = \varphi(-x),$$

which may be defined in generalized-function spaces in such a way that the corresponding operators act as in basic-function space when a generalized function is a basic function. We define the operators of (1.1) below in generalized-function space (f and g denote generalized functions, while $\varphi$ is a basic function):

$$(F[f],\ \varphi) = (f,\ F[\varphi]),\ (F^{-1}[f],\ \varphi) = (f,\ F^{-1}[\varphi]),$$
$$(\check{f},\ \varphi) = (f,\ \check{\varphi}),\ (D^m f,\ \varphi) = (-1)^m(f,\ D^m\varphi),$$
$$(T_y f,\ \varphi) = (f,\ T_{-y}\varphi),\ (f*\varphi)(x) = (f,\ T_{-x}\check{\varphi}),$$
$$(f*g,\ \varphi) = (g,\ \check{f} * \varphi).$$

By $\delta_y$ we denote a generalized function that acts in accordance with $(\delta_y, \varphi) = \varphi(y)$.

2. Consider the difference scheme

$$\sum_{|\alpha|\leqslant q_1} b^1_\alpha T_{\alpha h} f^{n+1} = \sum_{|\beta|\leqslant q_0} b^0_\beta T_{\beta h} f^n \tag{1.2}$$

in S' space. Here 0 $q_1$, $q_1 < \infty$; $b^1_\alpha$, $b^0_\beta$ are certain real constants and h is a parameter. The schemes of (1.2) may be written as

$$w_1 * f^{n+1} = w_0 * f^n, \tag{1.3}$$

where

$$w_j = \Lambda_j\delta_0 = \sum_{|\alpha|\leqslant q_j} b^j_\alpha \delta_{-\alpha h}, \quad j = 0,1.$$

Everywhere in what follows, we assume that the following are obeyed:

$$F[w_j](x) \neq 0 \ \forall x \in R,\ j = 0,1. \tag{1.4}$$

The $\Pi$ form of differential representation for the difference scheme of (1.2) that satisfies (1.4) is the equation

$$\frac{df(t)}{dt} = \frac{1}{\tau} F^{-1}[\ln F[w_0] - \ln F[w_1]] * f(t). \tag{1.5}$$

We consider the Cauchy problem for (1.2) and (1.5) with the initial condition

$$f^0 = f(0) = f_0. \tag{1.6}$$

The following assertion has been proved [5]:

THEOREM 1. Let $f_0 \in S'$; then the Cauchy problem of (1.2) and (1.6) and of (1.5) and (1.6) has a unique solution in space S'. Also: a) if f(t) is the solution to (1.5) and (1.6), then $\{f^n\}_{n=0}^{\infty}$ is the solution to (1.2) and (1.6), where $f^n = f(n\tau)$; b) if the solution to (1.2) and (1.6) is $\{f^n\}_{n=0}^{\infty}$, then there exists a one-parameter family of generalized functions f(t) that constitute a solution to the Cauchy problem of (1.5) and (1.6) and which is such that $f(n\tau) = f^n$.

It has also been shown [5] that the following applies in a certain generalized-function space dependent on the difference schemes of (1.2):

$$F^{-1}[\ln F[w_j]] * f = \sum_{k=0}^{\infty} \eta^j_k h^k (iD)^k f, \quad j = 0, 1. \tag{1.7}$$

Equation (1.7) applies in general in a weaker topology than that of S'. In (1.7) the $\eta^j_k h^k$ denote coefficients in the expansion of the function $\ln F[w_j](x)$ as a Taylor series near x = 0, i.e., $\ln F[w_j](x) = \sum_{k=0}^{\infty} \eta^j_k h^k x^k$.

3. We show that the right side in (1.7) can be taken as an asymptotic expansion of the generalized function $F^{-1}[\ln F[w_j]]*f$ for $h \to 0$ in topological space S'.

We recall that the formal series $\sum_{k=0}^{\infty} a_k h^k$ is termed an asymptotic expansion of $\varphi(h)$ for $h \to 0$ if for any integer N

$$\lim_{h\to 0}\left[\varphi(h)-\sum_{k=0}^{N} a_k h^k\right]\Big/h^N = 0.$$

This is written as

$$\varphi(h) \sim \sum_{k=0}^{\infty} a_k h^k \qquad (h \to 0).$$

In the same way we write the asymptoic expansions of the generalized functions dependent on the parameter: the series $\sum_{k=0}^{\infty} f_k h^k$ is called an asymptotic expansion of the one-parameter family of generalized functions $f(h)(f(h) \in S', f_k \in S' \forall k)$ for $h \to 0$ if for any basic function $\varphi \in S$ we have $(f(h), \varphi) \sim \sum_{k=0}^{\infty} h^k (f_k, \varphi) \quad (h \to 0)$.

It can be shown that if $f(h) \sim \sum_{k=0}^{\infty} f_k h^k \quad (h \to 0)$, than $F[f] \sim \sum_{k=0}^{\infty} F[f_k] h^k \ (h \to 0)$; moreover, for any given $x \in R$ (here and subsequently the subscript j is omitted) we have

$$\ln F[w](x) \sim \sum_{k=0}^{\infty} \eta_k x^k h^k \quad (h \to 0).$$

We show that for any basic function $\varphi \in S$ the product

$$\psi_N(h) = \left\{\ln F[w](x) - \sum_{k=0}^{N} \eta_k x^k h^k\right\}\varphi(x)/h^N$$

converges to zero for $h \to 0$ in S.

Since $\sum_{k=0}^{\infty} \eta_k x^k h^k$ converges to $\ln F[w](x)$ for $|xh| \leqslant r_0$ for some $r_0 > 0$, then

$$\left|\ln F[w](x) - \sum_{k=0}^{N} \eta_k x^k h^k\right| \leqslant C_0(h)\, h^N |x|^N \quad \text{for} \quad |xh| \leqslant r_0,$$

where $C_0(h) \to 0$ for $h \to 0$. Further, for $x \in R$

$$|\ln F[w](x)| \leqslant C_1(1+|xh|)^{M_1}$$

for certain constants $C_1$ and $M_1$, so $|\ln F[w](x)| \leqslant C_2 |xh|^{M_2}$ for $|xh| \geqslant r_0$, $M_2 > N$, and then

$$\left|\ln F[w](x) - \sum_{k=0}^{N} \eta_k x^k h^k\right| \leqslant \begin{cases} C_0(h)\, h^N |x|^N, & |xh| \leqslant r_0, \\ C h^M |x|^M, & |xh| \geqslant r_0. \end{cases}$$

Then $\forall \varphi(x) \in S$

$$\sup_{x\in R}\left|\left(\ln F[w](x) - \sum_{k=0}^{N} \eta_k x^k h^k\right)\varphi(x)/h^N\right| \to 0 \quad \text{for} \quad h \to 0.$$

Similarly one can show that

$$\sup_{x\in R}(1+|x|^2)^M\left|\frac{\partial^m}{\partial x^m}(\psi_N \varphi)\right| \to 0 \quad \text{for} \quad h \to 0.$$

Then for any $\varphi(x) \in S$ the one-parameter family $\varphi_N(h)$ converges to zero in S for $h \to 0$. Since for any generalized function $f \in S'$ we have

$$(\psi_N f, \varphi) = (f, \psi_N \varphi) \to 0 \quad \text{for} \quad h \to 0,$$

the following assertion is proved:

THEOREM 2. For any generalized function $f \in S'$

$$F^{-1}[\ln F[w]] * f \sim \sum_{k=0}^{\infty} \eta_k h^k (iD)^k f \quad (h \to 0).$$

in space S'.

Let $P_m^j = P_m^j(hx) = \sum_{k=0}^{m} \eta_k^j h^k x^k$; the equation

$$\frac{df}{dt} = \frac{1}{\tau} F^{-1}\left[P_m^0 - P_m^1\right] * f \tag{1.8}$$

is called the differential approximation to the difference scheme of (1.2).

Consider the following Cauchy problems:

$$\frac{df}{dt} = \frac{1}{\tau} F^{-1}[\ln F[w]] * f, \quad f|_{t=0} = f_0 \in S'; \tag{1.9}$$

$$\frac{df}{dt} = \frac{1}{\tau} F^{-1}[P_m] * f, \quad f|_{t=0} = f_0 \in S', \tag{1.10}$$

where $P_m = \sum_{k=0}^{m} \eta_k h^k x^k$; $\tau$ and h are parameters, and w is a finite linear combination of generalized functions $\delta_{\alpha h}$ that satisfies (1.4). In that case $\sum_{k=0}^{\infty} \eta_k h^k x^k$ converges to ln F[w](hx) in some region $\Omega \subset R$, i.e., for $xh \in \Omega$; let $\mathscr{R}_{\tau,h}(t)$, $\mathscr{R}_{m,\tau,h}(t)$ be the decision operators of (1.9) and (1.10), respectively.

It is clear that

$$\mathscr{R}_{\tau,h}(t) f = F^{-1}[(F[w])^{t/\tau}] * f, \ \mathscr{R}_{\tau,h}(t) : S' \to S' \tag{1.11}$$

(the latter follows from theorem 1);

$$\mathscr{R}_{m,\tau,h}(t) f = F^{-1}\left[e^{tP_m/\tau}\right] * f, \ \mathscr{R}_{m,\tau,h}(t) : S' \to Z'. \tag{1.12}$$

THEOREM 3. Let $f_{\tau,h}(t)$ be a solution to (1.9) and $f_{m,\tau,h}(t)$ be a solution to (1.10). Then $f_{\tau,h}(t) - f_{m,\tau,h}(t) = o(h^m)\,(h \to 0)$ in Z'.

Proof. It is necessary to show that $\forall \psi \in Z$

$$\lim_{h\to 0} ([f_{\tau,h}(t) - f_{m,\tau,h}(t)], \psi)/h^m = 0. \tag{1.13}$$

As $f_{\tau,h}(t) = \mathscr{R}_{\tau,h}(t) f_0$, $f_{m,\tau,h}(t) = \mathscr{R}_{m,\tau,h}(t) f_0$, we have from (1.11) and (1.12) that (1.13) is equivalent to

$$\lim_{h\to 0} \left(\left((F[w])^{t/\tau} - e^{tP_m/\tau}\right) g_0, \varphi\right)/h^m = 0 \quad \forall \varphi \in D(R), \tag{1.14}$$

where $g_0 = F[f_0]$; let $g(h; x) = h^{-m}(e^{\ln F[w] - P_m} - 1)$, and then for any integer nonnegative k and any compact K we have $D^k g(h; x) \to 0$ uniformly on K for $h \to 0$ and $\forall \varphi \in D(R)$; $g(h; x) \cdot \varphi(x) \to 0$ in D(R) for $h \to 0$. As $g_0 \in S'$ then $(gg_0, \varphi) \to 0$ for $h \to 0$ and $\forall \varphi \in D(R)$, and the latter means that

$$\lim_{h\to 0} \left(e^{P_m}\left(e^{\ln F[w] - P_m} - 1\right) g_0, \varphi\right)/h^m = \lim_{h\to 0} \left((F[w] - e^{P_m}) g_0, \varphi\right)/h^m = 0.$$

Therefore, (1.14) is proved and thus the assertion.

Using Theorems 1 and 3 we can prove:

Consequence. Let $f_0 \in S'$, $|f_h^n|_{n=0}^{\infty}$ be a solution to the Cauchy problem of (1.2) and (1.6) and $f_{m,\tau,h}(t)$ be a solution to the Cauchy problem of (1.8) and (1.6), and then

$$f_h^n - f_{m,\tau,h}(n\tau) = o\left(h^m\right) \quad (h \to 0)$$

in Z'.

So far we have considered the two-parameter family of (1.2), where the schemes are dependent on $\tau$ and h, and we have examined the behavior of the solution to the Cauchy problem as h tends to zero. In practice one usually supposes some relationship between $\tau$ and h. For example, for schemes such as (1.2) that approximate equations of hyperbolic type it is natural to put $\tau = \varkappa h$, $\varkappa$ = const, and we show that an assertion analogous to Theorem 3 applies when there is a connection between $\tau$ and h.

Let $\tau = \varkappa h^\alpha$, $0 < \alpha \leqslant l$, $\varkappa$ = const and let the expansion of the function ln F[w] as a series near zero have the first $l$ coefficients $\eta_0, \ldots, \eta_{l-1}$ also zero. Then $(1/\tau) \ln F[w](hx) = \frac{1}{\varkappa} x^\alpha \sum_{k=l}^{\infty} \eta_k (hx)^{k-\alpha}$. Since we have

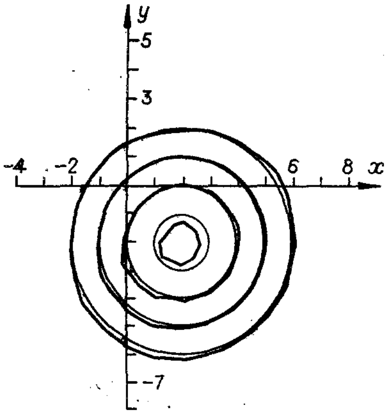


Fig. 1

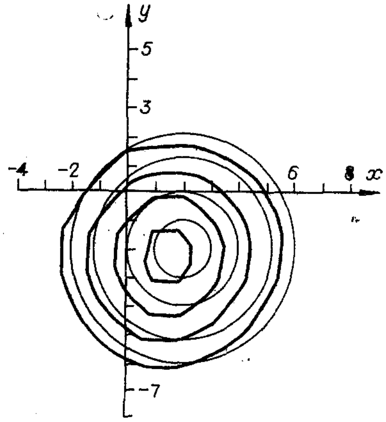


Fig. 2

$$\frac{1}{h^{m-\alpha}}\left[\frac{1}{\varkappa h^{\alpha}}\ln F[w](hx)-\frac{1}{\varkappa}x^{\alpha}\sum_{k=l}^{m}\eta_k(hx)^{k-\alpha}\right]\to 0$$

in D(R) for $h \to 0$, the following assertion applies:

THEOREM 4. Let $\tau = \varkappa h^{\alpha}$, $0 < \alpha \leqslant l$, $\alpha = \text{const}$, $\varkappa = \text{const}$, $\eta_0 = \eta_1 = \ldots = \eta_{l-1} = 0$, $f_h(t)$ be a solution to (1.9), and $f_{m,h}(t)$ be a solution to

$$\frac{df}{dt}=\frac{1}{\varkappa}F^{-1}\left[\sum_{k=l}^{m}\eta_k h^{k-\alpha}x^k\right]*f,\quad f|_{t=0}=f_0\in S',\quad m\geqslant l,$$

and then $f_h(t) - f_{m,h}(t) = o(h^{m-\alpha})$ for $h \to 0$ in Z'.

Using Theorems 1 and 4 we can prove:

Consequence. Let $\tau = \varkappa h^{\alpha}$, $0 < \alpha \leqslant l$, $f_0 \in S'$, $\{f_h^n\}_{n=0}^{\infty}$ be a solution to the Cauchy problem of (1.2) and (1.6), while $f_{m,h}(t)$ is the solution to the problem of (1.8) and (1.6). Let $h \to 0$ in such a way that $t = n\tau = \text{const}$. If $\eta_0^0 - \eta_0^1 = \eta_1^0 - \eta_1^1 = \ldots = \eta_{l-1}^0 - \eta_{l-1}^1 = 0$, then $f_h^n - f_{m,h}(n\tau) = o(h^{m-\alpha})$ for $h \to 0$ in Z'.

2. Invariant Difference Schemes. The concept of invariance in difference schemes has been introduced [6]. Numerical calculations for various models [1, 7-9] have shown that such schemes are better than ones that are not invariant but which reproduce the qualitative picture of the solution.

Consider the differential equation

$$u_t = Lu \tag{2.1}$$

where L is a linear differential operator with constant coefficients that contains only differentiation with respect to spatial variable x; this allows of the transformation group G [10], while T is an operator from G that maps the space of variables (t, x, u) into itself, i.e., $T(t, x, u) = (t', x', u')$, where $t' = t'(t, x, u)'$ $x' = x'(t, x, u)$; $u' = u'(t, x, u)$; if $u(t, x)$ satisfies (2.1), then the function $u'$, considered as a function of the variables $(t', x')$, also satisfies (2.1). Therefore, operator T acting in a finite-dimensional euclidian space generates some operator T' defined on the set of solutions to (2.1) and which converts one solution to another: $u' = T'u$.

Let two Cauchy problems be posed correctly for (2.1):

$$u|_{t=t_0} = \varphi(x); \tag{2.2}$$

$$u|_{t=t_0'} = \psi(x) \tag{2.3}$$

where $\varphi(x) = v(t, x)|_{t=t_0}$, $\psi(x') = v'(t', x')|_{t'=t_0'}$, where $v(t, x)$ is some solution to (2.1), $v' = T'v$. As the Cauchy problems of (2.1), (2.2), and (2.1), (2.3) are correct, the following applies to the solutions:

$$u' - T'u = 0, \tag{2.4}$$

where $u'(t', x')$ is the solution to the problem of (2.1) and (2.3) and $u(t, x)$ is the solution to (2.1) and (2.2).

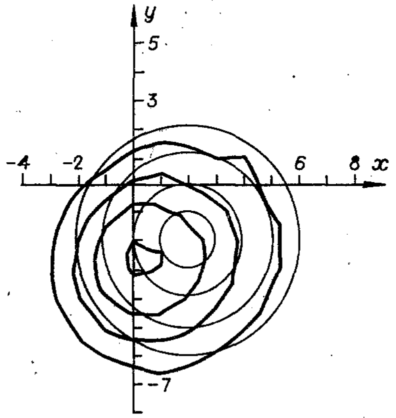


Fig. 3

2. Let (2.1) be approximated by the difference scheme

$$u^{n+1} = \Lambda u^n \tag{2.5}$$

with order of approximation k. We consider a second differential approximation for (2.5):

$$u_t = Lu + h^k L_1 u + h^{k+\beta} L_2 u,\ \beta > 0, \tag{2.6}$$

where $L_1$ and $L_2$ are certain differential operators with constant coefficients.

We formulate two Cauchy problems for (2.5) and (2.6) with the initial conditions

$$u^0 = u(0) = \varphi; \tag{2.7}$$

$$u^0 = u(0) = \psi, \tag{2.8}$$

where $\varphi$ and $\psi$ satisfy the above conditions (see (2.2) and (2.3)).

Let $u_h'$, $u_h$ be solutions to (2.5) and (2.8) or (2.5) and (2.7), respectively, while v and w are solutions to (2.6) and (2.7) or (2.6) and (2.8).

<u>THEOREM 5.</u> In general, $u_h' - T'u_h = O(h^k)$, but if (2.5) is invariant with respect to group G [6] then $u_h' - T'u_h = O(h^{k+\beta})$.

<u>Proof.</u> Let $v' = T'v$, and then $v'(t', x')$ satisfies

$$\frac{\partial v'}{\partial t'} = Lv' + h^k (L_1 v' + L_1' v') + h^{k+\beta}(L_2 v' + L_2' v'),$$

where $L_1'$ and $L_2'$ are certain operators dependent on group G.

This means that in Z' we have $w - v' = O(h^k)$ if $L_1' \not\equiv 0$ or $w - v' = O(h^{k+\beta})$ if $L_1' \not\equiv 0$ (the latter means that (2.5) is invariant with respect to G). Theorem 4 implies that $u_h' = w + o(h^{k+\beta})$ and $u_h = v + o(h^{k+\beta})$ in Z', so we finally get

$$u_h' - T'u_h = (u_h' - w) + (w - T'v) + T'v - T'u_h.$$

This means that $u_h' - T'u_h = O(h^k)$ if $L_1' \not\equiv 0$ (the scheme is not invariant) and $u_h' - T'u_h = O(h^{k+\beta})$ if $L_1' \not\equiv 0$ (the scheme is invariant).

3. A comparison has been made [9] of the results from various invariant and other schemes for the following model problem:

$$\frac{\partial u}{\partial t} = \alpha y \frac{\partial u}{\partial x} - \alpha x \frac{\partial u}{\partial y}; \tag{2.9}$$

$$u(0, x, y) = \begin{cases} 1 - \frac{1}{u_0} p, & p^2 = (x-a)^2 + (y-b)^2 \leqslant u_0^2, \\ 0, & p^2 > u_0^2. \end{cases} \tag{2.10}$$

The problem of (2.9) and (2.10) describes the rotation of a circular cone (height 1, radius of base $u_0$) around the origin with period $2\pi/\alpha$; (2.9) allows the rotational transformation. It is presented here to make a more detailed comparison of the numerical results from an invariant second-order splitting scheme and various standard second-order schemes, which were the Laks–Vendroff scheme (Richtmayer's modification [11]) and the MacCormack scheme [12].

The calculations showed that the invariant scheme reproduced the details of the exact solution more precisely. Figures 1-3 show the lines of $u(t, x, y) = c = \text{const}$ ($c = 0.2$; $0.4$; $0.6$; $0.8$) for the exact solution (circle) and the different solution at $t = 3$ derived from the invariant scheme (Fig. 1), the MacCormack scheme (Fig. 2), and the Laks—Vendroff scheme (Fig. 3). The calculations were performed with a rectangular net, $\Delta t/\Delta x = 0.01$, $\Delta t/\Delta y = 0.02$, $\Delta t$, $\Delta x$, $\Delta y$ being the steps in time and space, respectively.

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